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Quasi 2-metrics, 2-metrics and their decomposition

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Abstract

In this paper, we define the quasi semi 2-metric and quasi 2-metric functions, and after that we give some ways to decompose a 2-metric in two conjugated quasi 2-metrics.

Keywords: Quasi semi 2-metric; quasi 2-metric; decomposition.

1. Introduction

It is well known that, if one attempts to omit the requirement of symmetry in a semi metric or semi norm, then the structure of the space is drastically changed. Such unsymmetrical spaces have been studied by Wilson [1] who used the term quasi semi metric when the feature of symmetry is omitted in a semi metric. Many classical results of the analysis have been extended to such non-symmetric spaces like in [2], [3],[4].

In [5] J.M. Hernandez and C.H. Castaneda gave some ways to decompose metrics and norms. In this paper we will try to do the same for the 2-metric function.

Let us first give the definitions for the quasi semi 2-metrik and quasi 2-metric.

Definition 1.1

A quasi semi 2-metric function is a mapping: $\sigma : X \times X \times X \to \mathbb{R}^+$ such that:

(QSM1) $\forall x, y \in X$, $\exists z \in X$, so that $\sigma(x, y, z) \neq 0$,

(QSM2) $\sigma(x, y, z) = 0$ if at least two of three elements are equal.

If further (QSM3): $\sigma(x, y, z) = \sigma(x, z, y) = \sigma(z, y, x)$ holds, then σ is the *semi 2-metric function.*

Definition 1.2

A 2-metric is a mapping $\sigma : X \times X \times X \to \mathbb{R}^+$ such that:

1) $\sigma(x, y, z) \neq 0$
2) $\sigma(x, y, z) = 0$ when two of the three elements are equal
3) $\sigma(x, y, z) \leq \sigma(x, y, u) + \sigma(x, u, z) + \sigma(u, y, z)$
4) $\sigma(x, y, z) = \sigma(x, z, y) = \sigma(y, z, x)$ for every $x, y, z, u \in X$.

If we try to omit the requirement of the symmetry in definition 1.2 (the 4-rth property) we will obtain the concept of the quasi 2-metric:

Definition 1.3

A quasi 2-metric is a mapping $\rho : X \times X \times X \to \mathbb{R}^+$ such that the following properties hold:

1) $\rho(x, y, z) \neq 0$
2) $\rho(x, y, z) = 0$ when two of the three elements are equal

3) $\rho(x,y,z) \le \rho(x,y,u) + \rho(x,u,z) + \rho(u,y,z)$.

The pair (X,ρ) is called the quasi 2-metric space.

Example 1.4 $\mathbb{R}$ is a quasi 2-metric space, with $\rho(x,y,z) = \begin{cases} xy & if \quad xy \ge 0 \\ -\frac{xy}{2} & if \quad xy < 0 \end{cases}$.

A quasi semi 2-metric σ on X, has its conjugate $\bar{\sigma}$ defined as:

$$\bar{\sigma}(x,y,z) = \sigma(x,y,z) \text{ for every } x,y,z \in X.$$

It is easy to show that $\bar{\sigma}(x,y,z)$ is a quasi semi 2-metric. Mean while the function $\sigma^s : X \times X \to \mathbb{R}$ such that:

$$\sigma^s(x,y,z) = \max\{\sigma(x,y,z), \bar{\sigma}(x,y,z)\} \text{ is a semi 2-metric.}$$

Definition 1.5

If σ is a 2-metric on X and for some quasi 2-metric $\rho \ne \sigma$ is satisfied that $\rho^s = \sigma$ then the pair $(\rho,\bar{\rho})$ is called the *decomposition of the 2-metric* σ.

2. Main results

Theorem 2.1

Let (X,σ) be a 2-metric space. If $f : X \to \mathbb{R}$ is a function satisfying: $\sigma(u,y,z) \ge \sigma(x,y,z)$ and $\sigma(x,y,u) \ge \sigma(x,y,z)$ whenever $f(x) \le f(y) \le f(u)$, then there exists a decomposition of σ.

Proof.

Let: $\rho(x,y,z) = \rho_f(x,y,z) = \begin{cases} \sigma(x,y,z) & if \quad f(x) \le f(y) \\ 0 & if \quad f(x) > f(y) \end{cases}$.

Let us prove that ρ is a quasi 2-metric, we will start showing the triangle inequality by cases, for $x,y,z,u \in X$.

Case 1: If $f(x) > f(y)$

$$\rho(x,y,z) = 0 \le \rho(x,y,u) + \rho(x,u,z) + \rho(u,y,z).$$

Case 2: If $f(x) \le f(y)$. There are 3 subcases:

Subcase 1: If $f(x) \le f(y) < f(u)$

$$\rho(x,y,u) + \rho(x,u,z) + \rho(u,y,z) = \sigma(x,y,u) \ge \sigma(x,y,z) = \rho(x,y,z).$$

Subcase 2: If $f(x) \le f(u) \le f(y)$

$$\rho(x,y,z) = \sigma(x,y,z) \le \sigma(x,y,u) + \sigma(x,u,z) + \sigma(u,y,z)$$
$$= \rho(x,y,u) + \rho(x,u,z) + \rho(u,y,z).$$

Subcase 3: If $f(u) < f(x) \le f(y)$

$$\rho(x,y,u) + \rho(x,u,z) + \rho(u,y,z) = \sigma(u,y,z) \ge \sigma(x,y,z) = \rho(x,y,z).$$

In addition it is clear that $\rho(x,y,z) \ge 0$ and if $\rho(x,y,u) = \rho(x,u,z) = \rho(u,y,z) = 0$ then $\sigma(x,y,z) = 0$ that means: $x = y$ or $y = z$ or $x = z$.

Furthermore, $\bar{\rho}(x,y,z) = \rho(y,x,z) = \begin{cases} \sigma(y,x,z) & if \quad f(y) \le f(x) \\ 0 & if \quad f(y) > f(x) \end{cases}$, thus:

$$\rho^s(x,y,z) = \max\{\rho(x,y,z), \bar{\rho}(x,y,z)\} = \sigma(x,y,z). \blacksquare$$

Theorem 2.2

Let (X,σ) be a 2-metric space. If $card(X) \le \aleph_2$ then there exists a decomposition of σ.

Proof. If $card(X) \le \aleph_2$ then there exist $f : X \to \mathbb{R}$ an injective function. Given $k > 1$ we define:

$$\rho_k(x,y,z) = \rho_{f,k}(x,y,z) = \begin{cases} \sigma(x,y,z) & if \quad f(x) \ge f(y) \\ \frac{\sigma(x,y,z)}{k} & if \quad f(x) < f(y) \end{cases}.$$

It is clear that if: $\rho_k(x,y,z)=\rho_k(y,x,z)=0 \Rightarrow \sigma(x,y,z)=\dfrac{\sigma(y,x,z)}{k}=0$

or: $\sigma(y,x,z)=\dfrac{\sigma(x,y,z)}{k}=0 \Rightarrow x=y$.

For $x,y,z,u \in X$ we have the following cases:

Case 1: If $f(x)\geq f(y)\geq f(u)$

$$\begin{aligned}\rho_k(x,y,z)&=\sigma(x,y,z)\leq \sigma(x,y,u)+\sigma(x,u,z)+\sigma(u,y,z)\\&=\rho(x,y,u)+\rho(x,u,z)+k\rho(u,y,z)\\&\leq k\left(\rho(x,y,u)+\rho(x,u,z)+\rho(u,y,z)\right).\end{aligned}$$

Case 2: If $f(x)>f(u)>f(u)$

$$\begin{aligned}\rho_k(x,y,z)&=\sigma(x,y,z)\leq \sigma(x,y,u)+\sigma(x,u,z)+\sigma(u,y,z)\\&=\rho(x,y,u)+\rho(x,u,z)+\rho(u,y,z).\end{aligned}$$

Case 3: If $f(u)\geq f(x)\geq f(y)$

$$\begin{aligned}\rho_k(x,y,z)&=\sigma(x,y,z)\leq \sigma(x,y,u)+\sigma(x,u,z)+\sigma(u,y,z)\\&=\rho(x,y,u)+k\rho(x,u,z)+\rho(u,y,z)\\&\leq k\left(\rho(x,y,u)+\rho(x,u,z)+\rho(u,y,z)\right).\end{aligned}$$

Case 4: If $f(x)<f(y)\leq f(u)$

$$\begin{aligned}\rho_k(x,y,z)&=\frac{\sigma(x,y,z)}{k}\leq \frac{1}{k}\left[\sigma(x,y,u)+\sigma(x,u,z)+\sigma(u,y,z)\right]\\&=\frac{1}{k}\left[k\rho(x,y,u)+k\rho(x,u,z)+\rho(u,y,z)\right]\\&=\rho(x,y,u)+\rho(x,u,z)+\frac{1}{k}\rho(u,y,z)\\&\leq \rho(x,y,u)+\rho(x,u,z)+\rho(u,y,z).\end{aligned}$$

Case 5: If $f(x)<f(u)<f(y)$

$$\begin{aligned}\rho_k(x,y,z)&=\frac{\sigma(x,y,z)}{k}\leq \frac{1}{k}\left[\sigma(x,y,u)+\sigma(x,u,z)+\sigma(u,y,z)\right]\\&=\frac{1}{k}\left[k\rho(x,y,u)+k\rho(x,u,z)+k\rho(u,y,z)\right]\\&=\rho(x,y,u)+\rho(x,u,z)+\rho(u,y,z).\end{aligned}$$

Case 6: If $f(u)\leq f(x)<f(y)$

$$\begin{aligned}\rho_k(x,y,z)&=\frac{\sigma(x,y,z)}{k}\leq \frac{1}{k}\left[\sigma(x,y,u)+\sigma(x,u,z)+\sigma(u,y,z)\right]\\&=\frac{1}{k}\left[k\rho(x,y,u)+\rho(x,u,z)+k\rho(u,y,z)\right]\\&=\rho(x,y,u)+\frac{1}{k}\rho(x,u,z)+\rho(u,y,z)\\&\leq \rho(x,y,u)+\rho(x,u,z)+\rho(u,y,z).\end{aligned}$$

Because of the definition of ρ_k^s it is easy to show that $\rho_k^s=\sigma$.■

Theorem 2.3

Let (X,σ) be a 2-metric space with $card(X) \le \aleph_2$, let $k,l>1$ and $f,g: X \to \mathbb{R}$ be injective functions. Then $\rho_{f,k} \equiv \rho_{g,l}$ (2-metrically).

Proof. Let $x,y,z \in X$, we may assume that $k<l$. If $\rho_{f,k}$ and $\rho_{g,l}$ are given by:

$$\rho_{f,k} = \begin{cases} \sigma(x,y,z) & if \quad f(x)-f(y) \ge 0 \\ \dfrac{\sigma(x,y,z)}{k} & if \quad f(x)-f(y) < 0 \end{cases}$$

and:

$$\rho_{g,l} = \begin{cases} \sigma(x,y,z) & if \quad g(x)-g(y) \ge 0 \\ \dfrac{\sigma(x,y,z)}{l} & if \quad g(x)-g(y) < 0 \end{cases}.$$

<u>Case 1</u>: If $f(x)-f(y) \ge 0$ and $g(x)-g(y) \ge 0$, then $\rho_{f,k} = \sigma(x,y,z) = \rho_{g,l}$.

<u>Case 2</u>: If $f(x)-f(y) < 0$ and $g(x)-g(y) < 0$, then:

$$\rho_{f,k} = \frac{\sigma(x,y,z)}{k} > \frac{\sigma(x,y,z)}{l} = \rho_{g,l}.$$

<u>Case 3</u>: If $f(x)-f(y) \ge 0$ and $g(x)-g(y) < 0$, then:

$$\rho_{f,k} = \sigma(x,y,z) > \frac{\sigma(x,y,z)}{l} = \rho_{g,l}.$$

<u>Case 4</u>: If $f(x)-f(y) < 0$ and $g(x)-g(y) \ge 0$, then:

$$\rho_{f,k} = \frac{\sigma(x,y,z)}{k} > \sigma(x,y,z) = \rho_{g,l}.$$

And finally, if we perform a similar analysis to the Theorem 2.3, we obtain that for all $x,y,z \in X$:

$$k\rho_{f,k}(x,y,z) \ge \rho_{g,l}(x,y,z) \text{ and } l\rho_{g,l}(x,y,z) \ge \rho_{f,k}(x,y,z).$$

Hence:

$$k\rho_{f,k}(x,y,z) \ge \rho_{g,l}(x,y,z) \ge \frac{1}{l}\rho_{f,k}(x,y,z).$$

Therefore: $\rho_{f,k} \equiv \rho_{g,l}$.■

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